LARGE GAPS BETWEEN CONSECUTIVE PRIME NUMBERS CONTAINING SQUARE-FREE NUMBERS AND PERFECT POWERS OF PRIME NUMBERS

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Abstract. We prove a modification as well as an improvement of a result of K. Ford, D. R. Heath-Brown and S. Konyagin [2] concerning prime avoidance of square-free numbers and perfect powers of prime numbers.

1. Introduction

In their paper [2], K. Ford, D. R. Heath-Brown and S. Konyagin prove the existence of infinitely many "prime-avoiding" perfect k-th powers for any positive integer k.
They give the following definition of prime avoidance: an integer m is called prime avoiding with constant c, if $m+u$ is composite for all integers u satisfying[1]

$$|u| \leq c\,\frac{\log m \log_2 m \log_4 m}{(\log_3 m)^2}.$$

In this paper, we prove the following two theorems:

Theorem 1.1. *There is a constant $c>0$ such that there are infinitely many prime-avoiding square-free numbers with constant c.*

Theorem 1.2. *For any positive integer k, there are a constant $c=c(k)>0$ and infinitely many perfect k-th powers of prime numbers which are prime-avoiding with constant c.*

2. Proof of the Theorem 1.1

We largely follow the proof of [2].

Lemma 2.1. *For large x and $z \leq x^{\log_3 x/(10\log_2 x)}$, we have*

$$|\{n \leq x \,:\, P^+(n) \leq z\}| \ll \frac{x}{(\log x)^5},$$

where $P^+(n)$ denotes the largest prime factor of a positive integer n.

Proof. This is Lemma 2.1 of [2] (see also [8]). □

Lemma 2.2. *Let $\mathcal{R}$ denote any set of primes and let $a \in \{-1,1\}$. Then, for large x, we have*

$$|\{p \leq x \,:\, p \not\equiv a (\bmod r)\ (\forall r \in \mathcal{R})\}| \ll \frac{x}{\log x} \prod_{\substack{p\in\mathcal{R}\\ p\leq x}} \left(1-\frac{1}{p}\right).$$

Date: April 8, 2015.

[1] We denote by $\log_2 x = \log\log x$, $\log_3 x = \log\log\log x$, and so on.

Note. Here and in the sequel p will always denote a prime number.

Proof. This is Lemma 2.2 of [2] (see also [4]). $\square$

Lemma 2.3. *It holds*

$$\prod_{p\le w}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log w}\left(1+O\left(\frac{1}{\log w}\right)\right), \quad (w\to+\infty)\,,$$

where γ denotes the Euler-Mascheroni constant.

Proof. This is well known (cf. [5], p. 351). $\square$

Definition 2.4. *Let x be a sufficiently large number. Let also c_1 and c_2 be two positive constants, to be chosen later and set*

$$z=x^{c_1\log_3 x/\log_2 x},\ y=c_2\frac{x\log x\log_3 x}{(\log_2 x)^2}.$$

Let

$$\begin{aligned}
&P_1=\left\{p\ :\ p\le\log x\ \textit{or}\ z<p\le\frac{x}{4}\right\},\\
&P_2=\{p\ :\ \log x<p\le z\}\,,\\
&U_1=\{u\in[-y,y]\ :\ u\in\mathbb{Z},\ p\mid u\ \textit{for at least one}\ p\in P_1\},\\
&U_2=\{u\in[-y,y]\ :\ u\notin U_1,\ u\notin\{-1,0,1\}\},\\
&U_3=\{u\in U_2\ :\ |u|\ \textit{is a prime number}\},\\
&U_4=\{u\in U_2\ :\ |u|\ \textit{is composed only of prime numbers}\ p\in P_2\}.
\end{aligned}$$

Lemma 2.5. *We have*

$$U_2=U_3\cup U_4\,.$$

Proof. Assume that $u\in U_2\setminus U_4$. Then, by Definition 2.4, there is a prime number $p_0\notin P_2$ with $p_0\mid |u|$. Since by Definition 2.4 we know that $u\notin U_1$, we have

$$p_0>\frac{x}{4}.$$

Let p_1 be a prime with $p_1\mid\frac{|u|}{p_0}$. Then

$$|u|\ge p_0p_1>\frac{x}{4}\log x>y.$$

Thus, p_1 does not exist and we have $|u|=p_0$ and therefore $u\in U_3$. $\square$

Lemma 2.6. *We have*

$$|U_4|\ll\frac{x}{(\log x)^4}\,.$$

Proof. This follows immediately from Lemma 2.1. $\square$

Definition 2.7. *Let*

$$U_5=\{u\in U_3\ :\ p\nmid u+1\ \textit{for all}\ p\in P_2\}.$$

Lemma 2.8. *We can choose the constants c_1, c_2, such that*

$$|U_5|\le\frac{x}{3\log x}\,.$$

Proof. By Lemma 2.2, we have

$$|U_5| \ll \frac{y}{\log y} \prod_{p \in P_2} \left(1 - \frac{1}{p}\right).$$

Additionally, by Lemma 2.3, we have

$$\prod_{p \in P_2} \left(1 - \frac{1}{p}\right) = \frac{(\log_2 x)^2}{c_1 \log x \cdot \log_3 x} \left(1 + O\left(\frac{1}{\log_2 x}\right)\right).$$

Therefore,

$$|U_5| \ll \frac{c_2}{c_1} \frac{x}{\log x},$$

which proves Lemma 2.8 □

Definition 2.9. *We set*

$$U_6 = U_4 \cup U_5 \cup \{-1, 0, 1\}.$$

Lemma 2.10. *We have*

$$|U_6| \leq \frac{x}{2 \log x}.$$

Proof. This follows from Definition 2.9 and Lemmas 2.6, 2.8. □

Definition 2.11. *Let*

$$P_3 = \left\{p \,:\, \frac{x}{4} < p \leq x\right\}.$$

Let $\Phi : U_6 \to P_3$ be an injective map. Such a map Φ exists, since

$$|P_3| \geq |U_6|.$$

We denote

$$\Phi(u) = p_u.$$

We define

$$N = \prod_{p \leq \frac{x}{4}} p \prod_{u \in U_6} p_u.$$

We determine m_0 by the inequalities

$$1 \leq m_0 \leq N$$

and by the congruences:

$$\begin{aligned} &(1) \quad m_0 \equiv 0 \pmod p, \ (p \in P_1) \\ &(2) \quad m_0 \equiv 1 \pmod p, \ (p \in P_2) \\ &(3) \quad m_0 \equiv -u \pmod{p_u}, \ (p \in \Phi(U_6)). \end{aligned}$$

Lemma 2.12. *Let $m \geq 2y$, $m \equiv m_0 \pmod N$. Then $m + u$ is composite for $u \in [-y, y]$.*

Proof. If $u \in U_6$, then by the congruence (3) of Definition 2.11, we have

$$p_u \mid m_0 + u.$$

For $u \notin U_6$, by the definition of the sets $U_1, \ldots, U_5$, there is a $p \in P_1$, such that $p \mid u$ or there is a $p \in P_2$, such that $p \mid u + 1$. In both cases $p \mid m + u$, due to the congruences (1) and (2).

Thus, for all $u \in [-y, y]$ there is a prime p with $p \mid m + u$ and $p < m + u$. Hence, $m + u$ is composite for all $u \in [-y, y]$. □

Proof of Theorem 1.1. We now consider the arithmetic progression

$$(*) \qquad m = kN + m_0,\ k \in \mathbb{N}\,.$$

By elementary methods (see Heath-Brown [6] for references) the arithmetic progression (*) contains a square-free number

$$(1) \qquad m \leq N^{3/2+\varepsilon},$$

where $\varepsilon > 0$ is arbitrarily small.
By the prime number theorem, we have

$$(2) \qquad N \leq e^{x+o(x)}.$$

By Lemma 2.12, we know that $m+u$ is a composite number for $u \in [-y, y]$. By the estimates (1) and (2), we obtain

$$y \geq c\,\frac{\log m \log_2 m \log_4 m}{(\log_3 m)^2}$$

for a constant $c > 0$, which proves Theorem 1.1.

3. Sieve estimates

We introduce some notations borrowed with minor modifications form [2].
Let

$$\mathcal{A} = \text{a finite set of integers}$$

$$\mathcal{P} = \text{a subset of the set of all prime numbers}\,.$$

For each prime $p \in \mathcal{P}$, suppose that we are given a subset $\mathcal{A}_p \subseteq \mathcal{A}$.
Let $\mathcal{A}_1 = \mathcal{A}$,

$$P(z) = \prod_{\substack{p<z\\ p\in\mathcal{P}}} p$$

and

$$S(\mathcal{A}, \mathcal{P}, z) = \left| \mathcal{A} \setminus \bigcup_{p|P(z)} \mathcal{A}_p \right| .$$

Then for a positive square-free integer d composed of primes of $\mathcal{P}$ we define:

$$\mathcal{A}_d = \bigcap_{p|d} \mathcal{A}_p\,.$$

We assume that there is a multiplicative function $\omega(\cdot)$, such that for any d as above

$$|\mathcal{A}_d| = \frac{\omega(d)}{d}\,X + R_d\,,$$

for some R_d, where $X = |\mathcal{A}|$.
We set

$$W(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right)\,.$$

Lemma 3.1. (Brun's sieve)
Let the notations be as above. Suppose that:
1. $|R_d| \le \omega(d)$ *for any square-free integer* d *composed of primes of* $\mathcal{P}$
2. there exists a constant $A_1 \ge 1$*, such that*

$$0 \le \frac{\omega(p)}{p} \le 1 - \frac{1}{A_1}$$

3. there exist constants $\kappa > 0$ *and* $A_2 \ge 1$*, such that*

$$\sum_{w \le p < z} \frac{\omega(p) \log p}{p} \le \kappa \log \frac{z}{w} + A_2 \,, \text{ if } \; 2 \le w \le z \,.$$

Let b *be a positive integer and let* λ *be a real number satisfying*

$$0 < \lambda e^{1+\lambda} < 1 \,.$$

Then

$$S(\mathcal{A}, \mathcal{P}, z) \le XW(z) \left\{ 1 + 2 \, \frac{\lambda^{2b+1} e^{2\lambda}}{1 - \lambda^2 e^{2+2\lambda}} \, \exp\left((2b+3) \frac{c_1}{\lambda \log z} \right) \right\} + O\left(z^{2b + \frac{2.01}{e^{2\lambda/\kappa} - 1}} \right) .$$

Proof. This is part of Theorem 6.2.5 of [1]. □

4. Primes in arithmetic progressions

The following definition is borrowed from [7].

Definition 4.1. *Let us call an integer* $q > 1$ *a "good" modulus, if* $L(s, \chi) \ne 0$ *for all characters* $\chi \bmod q$ *and all* $s = \sigma + it$ *with*

$$\sigma > 1 - \frac{C_1}{\log\left[q(|t| + 1)\right]}.$$

This definition depends on the size of $C_1 > 0$.

Lemma 4.2. *There is a constant* $C_1 > 0$ *such that, in terms of* C_1*, there exist arbitrarily large values of* x *for which the modulus*

$$P(x) = \prod_{p < x} p$$

is good.

Proof. This is Lemma 1 of [7] □

Lemma 4.3. *Let* q *be a good modulus. Then*

$$\pi(x; q, a) \gg \frac{x}{\phi(q) \log x} \,,$$

uniformly for $(a, q) = 1$ *and* $x \ge q^D$.
Here the constant D *depends only on the value of* C_1 *in Lemma 4.2.*

Proof. This result, which is due to Gallagher [3], is Lemma 2 from [7]. □

5. Congruence conditions for the prime-avoiding number

Let x be a large positive number and y, z be defined as in Definition 2.4. Set

$$P(x) = \prod_{p \le x} p\,.$$

We will give a system of congruences that has a single solution m_0, with

$$0 \le m_0 \le P(x) - 1$$

having the property that the interval $[m_0^k - y, m_0^k + y]$ contains only few prime numbers.

Definition 5.1. *We set*

$$\begin{aligned}
\mathcal{P}_1 &= \{p \,:\, p \le \log x \quad \textit{or} \quad z < p \le x/40k\}\,,\\
\mathcal{P}_2 &= \{p \,:\, \log x < p \le z\}\,,\\
\mathcal{U}_1 &= \{u \in [-y, y],\, u \in \mathbb{Z},\, p \mid u \textit{ for at least one}\, p \in \mathcal{P}_1\}\,,\\
\mathcal{U}_2 &= \{u \in [-y, y] \,:\, u \notin \mathcal{U}_1\}\,,\\
\mathcal{U}_3 &= \{u \in [-y, y] \,:\, |u| \textit{ is prime}\}\,,\\
\mathcal{U}_4 &= \{u \in [-y, y] \,:\, P^+(|u|) \le z\}\,,\\
\mathcal{U}_5 &= \{u \in \mathcal{U}_3 \,:\, p \nmid u + 2^k - 1 \textit{ for } p \in \mathcal{P}_2\}
\end{aligned}$$

Lemma 5.2. *We have*

$$\mathcal{U}_2 = \mathcal{U}_3 \cup \mathcal{U}_4\,.$$

Proof. This is Lemma 2.5. □

Lemma 5.3. *We have*

$$|\mathcal{U}_4| \ll \frac{x}{(\log x)^4}\,.$$

Proof. This is Lemma 2.6. □

Lemma 5.4. *We can choose the constants c_1, c_2 such that*

$$|\mathcal{U}_5| \le \frac{x}{30k \log x}\,.$$

Proof. We have

$$\mathcal{U}_5 = \mathcal{U}_{5,1} \cup (-\mathcal{U}_{5,2})$$

with

$$\mathcal{U}_{5,1} = \{u \in \mathcal{U}_3 \,:\, p \nmid u + 2^k - 1 \text{ for } p \in \mathcal{P}_2\}$$

$$\begin{aligned}
\mathcal{U}_{5,2} &= \{u \in \mathcal{U}_3 \,:\, p \nmid -u + 2^k - 1 \text{ for } p \in \mathcal{P}_2\}\\
&= \{u \in \mathcal{U}_3 \,:\, p \nmid u - 2^k + 1 \text{ for } p \in \mathcal{P}_2\}\,.
\end{aligned}$$

We only give details for the estimate of $|\mathcal{U}_{5,1}|$, since the estimate of $|\mathcal{U}_{5,2}|$ is completely analogous.
We apply Lemma 3.1 with

$$\mathcal{A} = \{n \,:\, n \le y\}\,.$$

For $p \in \mathcal{P}_1$ we define $\mathcal{A}_p$ by

$$\mathcal{A}_p = \{n \in \mathcal{A} \,:\, n \equiv 0 \pmod{p} \text{ or } n \equiv 1 - 2^k \pmod{p}\}\,.$$

We check whether the conditions for the application of Lemma 3.1 are satisfied. For $d \mid P(y)$ we set:

$$\mathcal{A}_d = \bigcap_{p \mid d} \mathcal{A}_p \, .$$

We partition the interval $(0, y]$ into $\lfloor y/d \rfloor$ subintervals of length d and possibly one additional interval I_{last} of length less than d.
Let $\omega(d)$ be the number of the solutions $(\bmod\ d)$ of the system

$$\begin{aligned} &n \equiv 0 \pmod{p}, \quad p \in \mathcal{P}_1 \cup \mathcal{P}_2 \\ (**) \qquad &n \equiv 1 - 2^k \pmod{p}, \quad p \in \mathcal{P}_2 \, . \end{aligned}$$

By the Chinese Remainder Theorem, ω is multiplicative. Each interval of d consecutive integers contains exactly $\omega(d)$ solutions of the system (**).
Thus

$$\mathcal{A}_d = \frac{\omega(d)}{d}\, X + R_d,$$

where $|R_d| \leq \omega(d)$.
Thus, Lemma 3.1 is applicable and we obtain:

$$\begin{aligned} |\mathcal{U}_{5,1}| &\leq S(\mathcal{A}, \mathcal{P}, z) \\ &\ll y \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \prod_{\log x < p \leq z} \left(1 - \frac{2}{p}\right)\left(1 - \frac{1}{p}\right)^{-1} . \end{aligned}$$

Well known estimates of elementary prime number theory as in the proof of Lemma 2.8 in [2], give the result of Lemma 5.4. □

For the next definitions and results we follow the paper [2].

Definition 5.5. *Let*

$$\tilde{\mathcal{P}}_3 = \begin{cases} \left\{p \, : \, \frac{x}{40k} < p \leq x, \ p \equiv 2 \pmod{3}\right\} , & \textit{if } k \textit{ is odd} \\ \left\{p \, : \, \frac{x}{40k} < p \leq \frac{x}{2}, \ p \equiv 3 \pmod{2k}\right\} , & \textit{if } k \textit{ is even} , \end{cases}$$

We now define the exceptional set $\mathcal{U}_6$ as follows:
For k odd we set

$$\mathcal{U}_6 = \emptyset \, .$$

For k even and $\delta > 0$, we set

$$\mathcal{U}_6 = \left\{ u \in [-y, y] \, : \, \left(\frac{-u}{p}\right) = 1 \ \textit{ for at most } \frac{\delta x}{\log x} \textit{ primes } p \in \tilde{\mathcal{P}}_3 \right\} .$$

We shall make use of the following result from [2].

Lemma 5.6.

$$|\mathcal{U}_6| \ll_{\varepsilon} x^{1/2 + 2\varepsilon} \, .$$

Proof. This is formula (4) from [2], where $\mathcal{U}_6$ is denoted by $\mathcal{U}'$. □

Definition 5.7. *We set*

$$\mathcal{U}_7 = \mathcal{U}_4 \cup \mathcal{U}_5 \, .$$

Lemma 5.8. *We have*

$$|\mathcal{U}_7| \leq \frac{x}{20k \log x} \, .$$

Proof. This follows from Definition 5.7 and Lemmas 5.3, 5.4 □

We now introduce the congruence conditions, which determine the integer m_0 uniquely $(\bmod\ P(x))$.

Definition 5.9.

(C_1) $$m_0 \equiv 1 \pmod{p},\ \textit{for}\ p \in \mathcal{P}_1\,,$$

(C_2) $$m_0 \equiv 2 \pmod{p},\ \textit{for}\ p \in \mathcal{P}_2\,.$$

For the introduction of the congruence conditions (C_3) we make use of Lemma 5.8. Since

$$|\tilde{\mathcal{P}}_3| \geq |\mathcal{U}_7|,$$

there is an injective mapping

$$\Phi\ :\ \mathcal{U}_4 \to \tilde{\mathcal{P}}_3,\quad u \to \mathcal{P}_u.$$

We set

$$\mathcal{P}_3 = \Phi(\mathcal{U}_4)\,.$$

For all u, for which the congruence

$$m^k \equiv -(u-1) \pmod{p_u}$$

is solvable, choose a solution m_u of this congruence.
The set (C_3) of congruences is then defined by

(C_3) $$m_0 \equiv m_u \pmod{p_u},\quad p_u \in \mathcal{P}_3\,.$$

Let

$$\mathcal{P}_4 = \{p \in [0, x)\ :\ p \notin \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3\}\,.$$

The set of congruences is then defined by

(C_4) $$m_0 \equiv 1 \pmod{p},\quad p \in \mathcal{P}_4\,.$$

Lemma 5.10. *The congruence systems $(C_1) - (C_4)$ and the condition $1 \leq m_0 \leq P(x) - 1$ determine m_0 uniquely. We have $(m_0, P(x)) = 1$.*

Proof. The uniqueness follows from the Chinese Remainder Theorem. The coprimality follows, since by the definition of $(C_1) - (C_4)$ m_0 is coprime to all p, with $0 < p \leq x$. $\square$

Lemma 5.11. *Let $m \equiv m_0 \pmod{P(x)}$. Then $(m, P(x)) = 1$ and the number*

$$m^k + (u-1)$$

is composite for all $u \in [-y, y] \setminus \mathcal{U}_6$.

Proof. For $u \in \mathcal{U}_1$, there is $p \in \mathcal{P}_1$ with $p \mid u$. Therefore, since by Definition 5.9, the system (C_1) implies that $m_0 \equiv 1 \pmod{p}$, we have

$$m^k + (u-1) \equiv m_0^k + (u-1) \equiv 1 + u - 1 \equiv u \equiv 0 \pmod{p}\,,$$

i.e.

$$p \mid m^k + (u-1)\,.$$

For $u \in \mathcal{U}_3$, $u \notin \mathcal{U}_5$, there is $p \in \mathcal{P}_2$ with $p \mid u + 2^k - 1$.
Since by (C_2) $m_0 \equiv 2 \pmod{p}$, we have

$$m_0^k + (u-1) \equiv 2^k - 2^k \equiv 0 \pmod{p}\,,$$

i.e.

$$p \mid m^k + (u-1)\,.$$

The remaining cases, except $u \in \mathcal{U}_6$, are checked similarly. $\square$

6. Conclusion of the proof of Theorem 1.2

Let now x be such that $P(x)$ is a good modulus in the sense of Definition 4.1. By Lemma 4.2, there are arbitrarily large such elements x. Let D be a sufficiently large positive integer. Let $\mathcal{M}$ be the matrix with $P(x)^{D-1}$ rows and $U = 2\lfloor y \rfloor + 1$ columns, with the r, u element being

$$a_{r,u} = (m + rP(x))^k + u - 1,$$

where $1 \leq r \leq P(x)^{D-1}$ and $-y \leq u \leq y$.
Let $N(x,k)$ be the number of perfect k-th powers of primes in the column

$$\mathcal{C}_1 = \{a_{r,1} \,:\, 1 \leq r \leq P(x)^{D-1}\}\,.$$

Since $P(x)$ is a good modulus, we have by Lemma 4.2 that

$$N_0(x,k) \geq C_0(k)\,\frac{P(x)^{D-1}}{\log(P(x)^{D-1})}\,. \tag{5.1}$$

Let $\mathcal{R}_1$ be the set of rows R_1, in which these primes appear. We now give an upper bound for the number N_1 of rows $R_r \in \mathcal{R}_1$, which contain primes.
We observe that for all other rows $R_r \in \mathcal{R}_1$, the element

$$a_{r,1} = (m_0 + rP(x))^k$$

is a prime avoiding k-th power of the prime $m_0 + rP(x)$.

Lemma 6.1. *For sufficiently small c_2, we have*

$$N_1 \leq \frac{1}{2} N_0(x,k)\,.$$

Proof. For all v with $v - 1 \in \mathcal{U}_6$, let

$T(v) = \{r \,:\, 1 \leq r \leq P(x)^{D-1},\ m_0 + rP(x)$ and $(m_0 + rP(x))^k + v - 1$ are primes$\}$.

We have

$$N_1 \leq \sum_{v \in \mathcal{U}_6} T(v)\,. \tag{5.2}$$

For the estimate of $T(v)$ we apply again Lemma 3.1.
We set

$$\begin{aligned}
g(r) &= m_0 + rP(x)\\
h(r) &= (m_0 + rP(x))^k + v - 1\\
\mathcal{A} &= \{g(r)h(r) \,:\, 1 \leq r \leq P(x)^{D-1}\}\,,\\
\mathcal{A}_p &= \{n \in \mathcal{A} \,:\, n \equiv 0 \pmod{p}\}, \text{for each prime } p \text{ with } x < p \leq P(x)\,.
\end{aligned}$$

We let $\omega(p)$ be the number of solutions of the congruence

$$g(r)h(r) \equiv 0 \pmod{p}, \quad \text{for} \quad p > x\,.$$

Since $p \nmid P(x)$, the linear congruence

$$g(r) \equiv 0 \pmod{p}$$

has exactly one solution.
Let

$$\rho(p) = \left|\{n \pmod{p} \,:\, n^k + v - 1 \equiv\, 0 \pmod{p}\}\right|\,.$$

Then the congruence

$$h(r) \equiv 0 \ (\mathrm{mod}\ p)$$

has $\rho(p)$ solutions $(\mathrm{mod}\ p)$.
By Lemma 3.1, we have:

$$\begin{aligned} T(v) &\leq S(\mathcal{A}, \mathcal{P}, P(x)) \\ &\ll P(x)^{D-1} \prod_{x<p\leq P(x)} \left(1 - \frac{1}{p}\right) \prod_{x<p\leq P(x)} \left(1 - \frac{\rho(p)}{p}\right) . \end{aligned} \tag{5.3}$$

By Lemma 3.1 of [2], we have

$$\prod_{x<p\leq P(x)} \left(1 - \frac{\rho(p)}{p}\right) \ll_{k,\varepsilon} |v|^{\varepsilon} \frac{\log x}{\log P(x)} .$$

Lemma 6.1 now follows from (5.2), (5.3) and the bound for $|\mathcal{U}_6|$.
This completes the proof of Theorem 1.2. □

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